CIRCULAR PLATES UNDER THE ACTION OF DISCONTINUOUS LOADINGS

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In the papers [1 to 4] it is shown that in the case when the relative thickness (ε^2) of a circular plate is small, its behavior is similar to that of a membrane $(\varepsilon = 0)$ everywhere, except in a narrow portion near the boundary, where the "boundary layer" phenomenon takes place. However, similar phenomena may originate not only on the edge of the plate but also in the interior of it. In the present paper, with the aid of asymptotic methods developed for a symmetrically loaded circular plate in [3 and 4], it is established that an interior boundary layer [5] exists, if the loading is discontinuous in character. With the above in mind, asymptotic representations of solutions of problems are constructed and justified for evaluation of circular plates under the action of discontinuous loadings. This is illustrated by an example problem of a symmetically loaded circular plate which is under the action of a loading uniformly distributed along a certain circumference.

1. The system of von Kármán equations for the case of a symmetrically loaded circular plate, rigidly fixed along the edge, has the form

$$Av - \frac{u^2}{2} = 0, \qquad \varepsilon^2 Au + uv + \varphi(\rho) = 0 \tag{1.1}$$

$$A(\ldots) \equiv -\rho \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho} \rho(\ldots), \qquad u = \frac{dw}{d\rho}$$

$$\varepsilon^2 = \frac{h^2}{12(1-\sigma^2)a^2} \qquad \left(0 < \sigma < \frac{1}{2}\right), \qquad \varphi(\rho) = \frac{a}{Eh}\int_0^\rho q(t)\, t\, dt$$

$$u = 0, \ \ \frac{dv}{d\rho} - \frac{\sigma}{\rho} v = 0 \quad \text{for } \rho = 1; \quad \frac{u}{\rho} < \infty, \ \ \frac{v}{\rho} < \infty \quad \text{for } \rho = 0 \tag{1.2}$$

All the quantities, entering Equations (1.1) and (1.2), are dimensionless, in which wa is the deflection of the middle surface of the plate, vE/ρ is the radial stress, E is the Young's modulus, h is the plate thickness, a is the exterior radius, and $q(\rho)$ is the intensity of the normal loading.

In addition, it is assumed that the function $\varphi(\rho)$ and its derivatives to the $n+2$ order are piecewise continuous. Without loss of generality, we assume that $\varphi(\rho)$ has a unique jump at the point $\rho = b > 0$, i.e.

$$\varphi(b-0) \neq \varphi(b+0) \tag{1.3}$$

With these assumptions the following theorem can be formulated, using [6].

Theorem 1.1 The problem (1.1), (1.2) has unique solutions (v,u). The function v is nonnegative and twice continuously differentiable. The function u has continuous first and piecewise continuous second derivatives (finite jump at the point $\rho = b$)

2. For the solution of (1.1), (1.2) the following asymptotic representations are constructed:

$$v = \sum_{s=0}^{n+2} \varepsilon^s v_s + \sum_{s=0}^{n+2} \varepsilon^s h_s + \sum_{s=0}^{n+2} \varepsilon^s \xi_s + x_n$$
$$u = \sum_{s=0}^{n} \varepsilon^s u_s + \sum_{s=0}^{n} \varepsilon^s g_s + \sum_{s=0}^{n} \varepsilon^s \eta_s + z_n \tag{2.1}$$

The construction of the functions v_s, u_s and h_s, g_s is given in detail in [4]. To determine v_0, u_0 we had the system (membrane equations)

$$Av_0 - {}^1/_2 u_0^2 = 0, \qquad u_0 v_0 + \varphi(\rho) = 0 \tag{2.2}$$

with boundary conditions

$$\frac{dv_0}{d\rho} - \frac{\sigma}{\rho} v_0 = 0 \quad \text{for } \rho = 1, \qquad \frac{v_0}{\rho} < \infty \quad \text{for } \rho = 0 \tag{2.3}$$

and for the determination of v_s, u_s there is the system

$$Av_s - {}^1/_2 \sum_{k+j=s} u_k u_j = 0, \qquad \sum_{k+j=s} u_k v_j + Au_{s-2} = 0 \tag{2.4}$$
$$(s = 1, 2, \ldots, n+2; \quad u_{-1} = 0)$$

with the boundary conditions

$$\frac{v_s}{\rho} < \infty \quad \text{for } \rho = 0, \qquad \frac{dv_s}{d\rho} - \frac{\sigma}{\rho} v_s = B_s \quad \text{for } \rho = 1 \tag{2.5}$$

Here B_s are found by equating to zero the coefficients of ε^s in Expression

$$\sum_{s=0}^{n+2} \varepsilon^s \left[B_s + \frac{dh_s}{d\rho} - \frac{\sigma}{\rho} h_s \right] = 0 \qquad \text{for } \rho = 1$$

Functions of the boundary layer type h_s, g_s, which compensate for the mismatch of the functions satisfying the boundary conditions (1.2), are defined from the differential equations with constant coefficients

$$\frac{d^2 h_i}{dt^2} = 0 \qquad (i = 0, 1) \tag{2.6}$$

$$\frac{d^2h_{s+2}}{dt^2} = R_1 h_{s+1} + R_2 h_s - \sum_{k+j+l=s} t^l (1-t) u_{kl} g_j - \frac{1}{2} \sum_{i+j=s} (1-t) g_i g_j \quad (2.7)$$

$$\frac{d^2 g_s}{dt^2} - v_{00} g_s = R_1 g_{s-1} + R_2 g_{s-2} + \sum_{\substack{k+j+l=s \\ (s\neq j)}} t^l (1-t) v_{kl} g_j -$$

$$- \sum_{j+m=s} t^l (1-t) g_j h_m + \sum_{k+m+l=s} t^l (1-t) u_{kl} h_m$$

with boundary conditions

$$g_s|_{t=0} = -u_{s0}, \qquad g_s|_{t=\infty} = 0, \qquad h_s'|_{t=\infty} = 0 \qquad (2.8)$$

Here

$$R_1(\ldots) \equiv 2t \frac{d^2(\ldots)}{dt^2} + \frac{d(\ldots)}{dt}, \qquad R_2(\ldots) \equiv -t^2 \frac{d^2(\ldots)}{dt^2} - t\frac{d(\ldots)}{dt} + (\ldots)$$

$$g_{-2} = g_{-1} = 0, \quad v_{00} = \frac{1}{1-\sigma} \int_0^1 \eta \int_\eta^1 \frac{\varphi^2}{\xi v_0^2} d\xi\, d\eta > 0, \quad s = 0, 1, \ldots, n$$

$$v_k = \sum_{l=0} v_{kl} (1-\rho)^l, \qquad u_k = \sum_{l=0} u_{kl} (1-\rho)^l$$

$v_k(\rho)$ and $u_k(\rho)$ are the expansions in Taylor's series at the point $\rho = 1$.

But in [1 to 4] the investigations were conducted for the case of sufficiently smooth loadings $\varphi(\rho)$. As is evident from (1.3), here this condition is violated. We will show that the discontinuity of $\varphi(\rho)$ at the point $\rho = b$ produces in the neighborhood of this point the phenomenon of the interior boundary layer [5]. Two theorems are necessary in what follows.

T h e o r e m 2.1. The problem (2.2), (2.3) has unique solutions (v_0, u_0). The function v_0 and its first derivative are continuous and the estimate

$$v_0(\rho) > \rho \frac{1+\sigma}{2(1-\sigma)} \int_0^1 x\, dx \int_x^1 \frac{\varphi^2}{y v_0^2} dy > 0 \qquad (2.9)$$

is valid.

All the following derivatives of the function v_0, and likewise the function u_0 and its first drivative are piecewise continuous (they have finite jumps at the point $\rho = b$)

The proof of the theorem almost literally coincides with the proof of Theorem 2.2 in [7]. It thereby becomes obvious that v_0 appears as the limit of the sequence determined by the relations

$$v_{n+1} = v_n - \delta_n \qquad (n = 1, 2, \ldots)$$

$$v_1 = A^{-1}\left(\frac{\varphi^2}{2C^2}\right), \qquad C = \max\left[\frac{\varphi^2(\rho)}{\rho}\right]^{1/2} \qquad (0 \leqslant \rho \leqslant 1)$$

where δ_n is the solution of Equation

$$\frac{1}{\rho} A\delta_n + M\delta_n - \alpha_n = 0, \qquad \left[\frac{\delta_n}{\rho}\right]_{\rho=0} < \infty, \qquad \left[\frac{d\delta_n}{d\rho} - \frac{\sigma}{\rho}\delta_n\right]_{\rho=1} = 0$$

$$\alpha_n = \frac{1}{\rho} A v_n - \frac{\varphi^2}{2\rho v_n^2}, \qquad M = \max\left|\frac{\varphi^2}{\rho v_1^3}\right| \qquad (0 \leqslant \rho \leqslant 1)$$

Theorem 2.2. Problem (2.4), (2.3) has a unique solution v_s, u_s ($s = 1, 2, \ldots$). The function v_s and its first derivative are continuous, while the higher derivatives of v_s, and also the function u_s together with its derivatives are piecewise continuous (they have finite jumps at the point $\rho = b$).

Theorem 2.2 follows as a consequence of theorem 4 from [4] and theorem 2.1 of the present paper.

Applying Theorems 1.1, 2.1 and 2.2, we note that the differences

$$v^n = v - \sum_{s=0}^{n+2} \varepsilon^s (v_s + h_s), \qquad u^n = u - \sum_{s=0}^{n} \varepsilon^s (u_s + g_s)$$

and their derivatives have finite jumps at the point $\rho = b$. Indeed, while the function $u(\rho)$ is continuously differentiable at the point $\rho = b$, the functions $u_s(\rho)$ ($s = 0, 1, \ldots$) together with their derivatives are discontinuous at this point. Further, the differences v^n and u^n in the neighborhood of $\rho = b$ have the character of a boundary layer. In order to find this character we introduce the functions ξ_s and η_s which are sought in the form

$$v^n = \sum_{i=0}^{n} \varepsilon^i \xi_{ki}, \qquad u^n = \sum_{i=0}^{n} \varepsilon^i \eta_{ki} \quad (k = 1, 2) \tag{2.10}$$

Here

$$\xi_i = \xi_{1i}, \quad \eta_i = \eta_{1i} \quad \text{for } \rho < b, \qquad \xi_i = \xi_{2i}, \quad \eta_i = \eta_{2i} \quad \text{for } \rho > b$$

We let further, $r = |b - \rho|$ and

$$v_k = v_{k0} + v_{k2} r + \ldots + v_{kn} r^n, \quad u_k = u_{k0} + u_{k1} r + \ldots + u_{kn} r^n \tag{2.11}$$

which are the corresponding expansions in Taylor's series at the point $r = 0$. Now we substitute (2.10) and (2.11) into (1.1), and perform the substitution $r = \varepsilon t$ and equate to zero the coefficients of $\varepsilon^0, \varepsilon^1, \ldots \varepsilon^n$. We obtain the system (2.6) to (2.8) and (2.11) for the determination of ξ_{ks} and η_{ks} with the substitutions h_s by ξ_{ks}, g_s by η_{ks} and $v_0(1)$ by $v_0(b)$. The unknown boundary conditions at $t = 0$ for η_{ks} ($k = 1, 2$) remain unknown. Applying Theorems 1.1, 2.1 and 2.2 we conclude that the missing boundary conditions are determined from the requirement that the sum

$$(u_0 + \eta_0) + \varepsilon(u_1 + \eta_1) + \ldots + \varepsilon^n (u_n + \eta_n)$$

must be continuous together with its derivative. Then, if we introduce the notation

$$[F] = F(b + 0) - F(b - 0) \tag{2.12}$$

the condition of continuity can be written as

$$[u_s + \eta_s] = \left[\frac{\partial}{\partial \rho}(u_s + \eta_s)\right] = 0 \qquad (s = 0, 1, \ldots, n) \tag{2.13}$$

Further, from (2.6) we obtain that $\xi_0 = \xi_1 = 0$ This corresponds to the condition that the difference $v - v_0$ and its first derivative are continuous at the point $\rho = b$.

Now from (2.7) for $s = 0$ we obtain

$$\frac{d^2\eta_{k0}}{dt^2} - v_0(b)\,\eta_{k0} = 0, \qquad \eta_{k0}|_{t=\infty} = 0 \qquad (k = 1, 2) \tag{2.14}$$

Hence we find that

$$\eta_0 = C_1 \exp\left(-\sqrt{v_0(b)}\,\frac{b-\rho}{\varepsilon}\right) \quad \text{for } \rho < b$$
$$\eta_0 = C_2 \exp\left(-\sqrt{v_0(b)}\,\frac{\rho-b}{\varepsilon}\right) \quad \text{for } \rho > b \tag{2.15}$$

In order to determine the constants C_k we substitute (2.15) into (2.13) for $s = 0$ and we obtain a system of two linear algebraic equations for C_1 and C_2 . Solving this system we find

$$C_1 = \frac{1}{2}\left([u_0] + \frac{\varepsilon}{\sqrt{v_0(b)}}\left[\frac{\partial u_0}{\partial \rho}\right]\right), \qquad C_2 = -\frac{1}{2}\left([u_0] - \frac{\varepsilon}{\sqrt{v_0(b)}}\left[\frac{\partial u_0}{\partial \rho}\right]\right) \tag{2.16}$$

The functions η_s ($s = 1, 2, \ldots$) are determined in an analogous form from the equations of the form (2.14), but being nonhomogeneous, and the functions ξ_s are determined from Formulas (2.7) by repeated integrations. It is not difficult to see that the functions ξ_s and η_s are functions of the boundary layer type [5].

3. For the foundations of the asymptotic representations we proceed from the following Lemma.

L e m m a 3.1 . Let $\varphi_k = v - x_k$ and $\psi_k = u - z_k$. Then in each interval $[0, b]$ and $[b, 1]$ the estimates

$$A\varphi_k - {}^1\!/_2\psi_k^{\,2} = O(\rho\varepsilon^{k+1}), \qquad \varepsilon^2 A\psi_k + \varphi_k\psi_k + \varphi(\rho) = O(\rho\varepsilon^{k+1}) \tag{3.1}$$

are valid.

This lemma follows from Lemma 3 of [4], applied separately in the intervals $[0, b]$ and $[b, 1]$.

L e m m a 3.2 . For sufficiently small ε $(0 < \varepsilon < \varepsilon_1)$ for all $\rho \in [0, 1]$ the following relations are valid:

$$1)\ \ \varphi_k \geqslant 0, \qquad 2)\ \ \min\frac{\varphi_k}{\rho} > \frac{T}{2}, \qquad T = v_0(1) > 0 \tag{3.2}$$

The inequalities (3.2) are easily obtained as a consequence of Lemma 5 of [4], Theorem 2.1, and (2.6), (2.9).

L e m m a 3.3 . For x_k and z_k the following energy estimation is valid:

$$\left(\frac{1}{2} - \sigma\right)\int_0^1 \left(\frac{dx_k}{d\rho}\right)^2 d\rho + \frac{1}{2}\int_0^1 \frac{x_k^{\,2}}{\rho^2}\,d\rho + \varepsilon^2\int_0^1 \left(\frac{dz_k}{d\rho}\right)^2 d\rho + \frac{\varepsilon^2}{2}\int_0^1 \frac{z_k^{\,2}}{\rho^2}\,d\rho +$$
$$+ \frac{T}{4}\int_0^1 z_k^{\,2}\,d\rho \leqslant C\varepsilon^{k+1}\int_0^1 (|x_k| + |z_k|)\,d\rho, \qquad T = v_0(1) > 0 \tag{3.3}$$

We begin from considering the interval $[0, b]$. We substract (3.1) from (1.1) and multiply the first difference by $(v - \varphi_k)/\rho$, and the second by $(u - \psi_k)/\rho$, and integrate from 0 to 1 and sum the results. We perform an analogous operation in the interval $[b, 1]$ and the result obtained is added to the previous one. The result of these operations is

$$\int_0^1 \left(\frac{dx_k}{d\rho}\right)^2 d\rho + \frac{1}{2}\int_0^1 \frac{x_k^2}{\rho^2}\, d\rho + \varepsilon^2 \int_0^1 \left(\frac{dz_k}{d\rho}\right)^2 d\rho + \frac{\varepsilon^2}{2}\int_0^1 \frac{z_k^2}{\rho^2}\, d\rho +$$
$$+\int_0^1 \frac{(\varphi_k + v)}{\rho} z_k^2 d\rho - \left(\sigma + \frac{1}{2}\right) x_k^2(1) - \left[x_k \frac{dx_k}{d\rho} + \frac{1}{2} x_k^2\right]_{b-0}^{b+0} - \qquad (3.4)$$
$$- \varepsilon^2 \left[z_k \frac{dz_k}{d\rho} + \frac{1}{2} z_k^2\right]_{b-0}^{b+0} \leqslant C\varepsilon^{k+1}\int_0^1 (|x_k| + |z_k|)\, d\rho$$

Let us show that the nonintegrated terms appearing in the square brackets are equal to zero. Obviously, that for this to be true it is necessary to demonstrate that x_k and z_k are continuous together with their first derivatives at the point $\rho = b$. For the function $z_k(\rho)$ this follows from the smoothness of $u(\rho)$ by virtue of Theorem 1.1 and the smoothness of $\psi_k(\rho)$ by virtue of conditions (2.13). For the function $x_k(\rho)$ this follows from the smoothness of $v(\rho)$ and $v_s(\rho)$ $(s = 0,1,\ldots)$ by virtue of Theorems 1.1, 2.1 and 2.2 and the fact that the ξ_s are obtained by the double integration of expressions having possible finite j mps at the point $\rho = b$. So, the expressions in the square brackets are equal to zero, and the inequality (3.3) follows from (3.4) with the aid of Theorem 1.1, Lemma 3.2 and the simple inequality

$$v^2(1) = \left(\int_0^1 \frac{dv}{d\rho} d\rho\right)^2 \leqslant \int_0^1 \left(\frac{dv}{d\rho}\right)^2 d\rho$$

T h e o r e m 3.1. Let the function $\varphi(\rho)$ satisfy condition (1.3) and for each of the intervals $[0, b]$ and $[b, 1]$ it has $n+2$ continuous derivatives. Then the asymptotic representation (2.1) holds, in which, the estimated remainder allowed is

$$\begin{aligned}
&\max_\rho |x_n(\rho)| \leqslant m_1\varepsilon^{n+1} \quad (n \geqslant 0), && \max_\rho |z_n(\rho)| \leqslant m_2\varepsilon^{n+1/2} \quad (n \geqslant 0)\\
&\max_\rho \left|\frac{dx_n}{d\rho}\right| \leqslant m_3\varepsilon^{n+1} \quad (n \geqslant 0), && \max_\rho \left|\frac{dz_n}{d\rho}\right| \leqslant m_4\varepsilon^{n-1} \quad (n \geqslant 2) \qquad (3.5)\\
&\max_\rho \left|\frac{d^2x_n}{d\rho^2}\right| \leqslant m_5\varepsilon^{n-1/2} \quad (n \geqslant 1), && \max_\rho \left|\frac{d^2z_n}{d\rho^2}\right| \leqslant m_6\varepsilon^{n-2} \quad (n \geqslant 3)
\end{aligned}$$
$$(0 \leqslant \rho \leqslant 1)$$

4. In the case of other boundary conditions, for instance, free clamping or simply supporting the principal term of the interior boundary layer will be of the form (2.14) to (2.16). Whereby, the exponential character of the boundary layer can be explained by the fact that the radial force in the interior points of the membrane is positive (see Lemma 1 of [4]). If, however, one can construct the following approximations of the degenerate problem analogous to (2.4), (2.5), then the subsequent asymptotic representation can be constructed with the aid of the equations of the form (2.6) and (2.7).

5. Example. Let a circular plate rigidly fixed along the contour be under the action of a symmetrical loading of intensity p, uniformly distributed along some circumference of radius $b > 0$. (The problem is formulated in [1], page 168). To further define the problem we let $b^2 = 0.5$, $\sigma = 0.3$, $a/h \approx 8.704$, and $q = (a/Eh)p$.

Then the equilibrium state of the plate is described by equations (1.1) and (1.2) in which

$$\varphi(\rho) = 0 \quad \text{for} \quad 0 \leqslant \rho < b, \qquad \varphi(\rho) = qb \quad \text{for} \quad b \leqslant \rho \leqslant 1 \tag{5.1}$$

Without loss of generality it can be assumed that

$$\varphi(\rho) = 0 \quad \text{for} \quad 0 \leqslant \rho < b, \qquad \varphi(\rho) = 1 \quad \text{for} \quad b \leqslant \rho \leqslant 1 \tag{5.2}$$

since the problem (1.1), (1.2), (5.1) reduces the problem (1.1), (1.2), (5.2) with the simple substitutions

$$v = \alpha\,(qb)^{2/3}, \qquad u = \beta\,(qb)^{1/3}, \qquad \varepsilon_1{}^2 = \varepsilon^2\,(qb)^{-2/3} \tag{5.3}$$

It is not difficult to calculate that the relative thickness of the plate $\varepsilon = 0.035$, and therefore, the solution of the problem can be constructed with the aid of the asymptotic representation (2.1).

The fundamental difficulty in the construction of the asymptotic representation is the solution of the problem (2.2), (2.3). This problem could be solved by making use of the algorithms given in Theorem 2.1. But in the case of the function $\varphi(\rho)$ specified in Formulas (5.2), it is more convenient to take advantage of the method of power series. For this purpose we eliminate u_0 from (2.2), (2.3) and perform the substitutions

$$p_0 = \rho v_0, \qquad \rho^2 = 1 - x \tag{5.4}$$

Making use of (5.2) the results are

$$-8p_0{}^2\, d^2p_0/dx^2 - 1 = 0 \quad \text{for} \quad 0 \leqslant x \leqslant b^2 \tag{5.5}$$

$$d^2p_0/dx^2 = 0 \quad \text{for} \quad b^2 \leqslant x < 1 \tag{5.6}$$

$$p_0|_{x=1} = 0, \qquad [2\,dp_0/dx + (1 + \sigma)\,p_0]_{x=0} = 0 \tag{5.7}$$

The solution of problem (5.5) to (5.7) in the interval $[0, b^2]$ is approximated by a segment of the power series

$$P_n(x) = a_0 + a_1x + \ldots + a_nx^n \qquad (n = 2, 3, \ldots) \tag{5.8}$$

In order to determine the constants a_s we substitute (5.8) into (5.5) and into the second boundary condition of (5.7) and then we equate to zero the coefficients with various powers of x. The resulting relations are

$$a_1 = -\frac{1+\sigma}{2}\,a_0, \qquad a_2 = -\frac{1}{16a_0{}^2} \tag{5.9}$$

$$a_s = -\frac{1}{s\,(s-1)\,a_0{}^2} \sum_{\substack{k+m+t=s+2 \\ (t\geqslant 2,\ t\neq s)}} t\,(t-1)\,a_k a_m a_t \tag{5.10}$$

From (5.9) and (5.10) we find

$$a_s = -\frac{1}{a_0{}^2} \sum^{[1/3 s-1]} b_k{}^{(s)} \left(\frac{1}{a_0}\right)^{3k} \qquad (s = 3, 4, 5, \ldots) \tag{5.11}$$

Here the $b_k{}^{(s)}$ are completely determined numbers for for a given value σ. The table below gives several values of $b_k{}^{(s)}$ for $\sigma = 0.3$, employed in what follows.

	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$	$s=9$
$-b_0^{(s)}\ 10^1$	0.66666	0.4	0.256	0.17066	0.11702	0.13457	0.13516
$-b_1^{(s)}\ 10^2$		0.52083	0.91666	1.1333	1.2114	1.19768	1.09453
$-b_2^{(s)}\ 10^3$				0.47742	1.4484	2.72318	3.36034
$-b_3^{(s)}\ 10^4$						0.56577	2.12603

From (5.11) it follows that in order to determine the values of a_s it is necessary to find a_0. We note first that the solution of problem (5.5) to (5.7) in the interval $[b^2, 1]$ has the form

$$p_0 = C\,(1 - x) \qquad (b^2 \leqslant x \leqslant 1) \tag{5.12}$$

Here C is a certain constant. In order to find the constant C and together with it a_0, we take advantage of Theorem 2.1 concerning the continuity of the function v_0 and its first derivative. This, together with (5.8) and (5.12) leads to the following relations at the point $x = b^2$:

$$\sum_{s=0}^{n} a_s b^{2s} = C\,(1 - b^2), \qquad \sum_{s=0}^{n} s a_s b^{2(s-1)} = -C \tag{5.13}$$

Eliminating C we deduce from (5.13)

$$\sum_{s=0}^{n} a_s b^{2(s-1)}(b^2 + s\,(1 - b^2)) = 0 \tag{5.14}$$

Applying (5.11), we obtain from (5.14) the following algebraic equation with respect to $z = a_0^3$:

$$f_m(z) = z^m + c_1 z^{m-1} + \ldots + c_{m-1} z + c_m = 0 \qquad (z = a_0^3) \tag{5.15}$$

Now if in (5.8) we take the value $n = 2(k + 1)$, then the order of Equation (5.15) will be equal to k

In order to select amongst the roots of $f_m(z)$ the necessary root, we observe that $a_0 = v_0(1) > 0$ (see Theorem 2.1). But Equation (5.15) has a unique positive root. This follows from the fact, that all c_i $(i = 1, 2, \ldots, m)$ are negative according (5.11), and then uniqueness follows from Descartes theorem concerning the number of positive roots of a polynomial. We note that the positive root of Equation $f_m(z) = 0$ is conveniently found by Newton's method, in which the initial approximation is taken equal to the upper bound of the positive roots of the polynomial determined according to the Maclaurin method, i.e.

$$z_0 = 1 + \max_i |c_i| \qquad (1 \leqslant i \leqslant m)$$

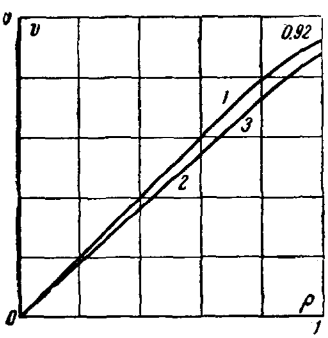

Fig.1

Finally, having a_0 determined, we find the a_s $(s = 1, 2, \ldots)$ according to Formulas (5.9), (5.11), and the constant C is found from any of Formulas (5.13). With the method described above for the values $\sigma = 0.3$ and $b^2 = 0.5$ the approximate solution of the problem (2.2), (2.3), (5.2) was obtained. For the approximation of p_0 the polynomials P_7 and P_9 were constructed (see (5.8)). With this it is useful to note the satisfaction of the inequality

$$\max_x |P_7(x) - P_9(x)| \leqslant 0.002 \qquad (0 \leqslant x \leqslant {}^1\!/_2) \tag{5.16}$$

Now, applying (5.4) and relations (2.2), we compute the displacements v_0, u_0. The deflection of points of the middle surface of the membrane are found from Formula

$$w_0 = \int_1^\rho u_0 \, d\rho \qquad (w_0(1) = 0) \tag{5.17}$$

The graphs of the functions v_0, u_0 and w_0 are represented, respectively,

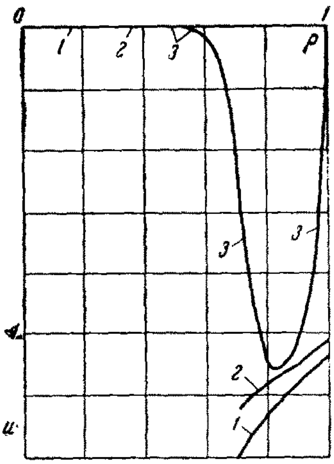

Fig. 2

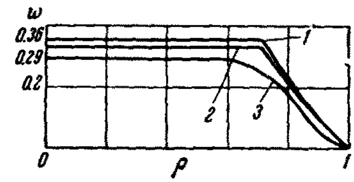

Fig. 3

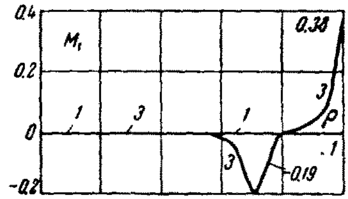

Fig. 4

in Figs. 1, 2 and 3 and marked with number 1. We note that the graph of u_0 has a discontinuity at the point $\rho = b$, where $u_0(\rho) = 0$ for $0 \leqslant \rho < b$. Further, from (2.6), (2.7) for $s = 0$ we find g_0 and h_2, and from (2.14) to (2.16) we determined η_0. For the determination of v_1 and u_1, from (2.4) and (2.5) we obtain

$$A v_1 + \frac{u_0^2}{v_0} v_1 = 0, \qquad u_1 = -\frac{u_0 v_1}{v_0} \tag{5.18}$$

$$\left[\frac{v_1}{\rho}\right]_{\rho=0} < \infty, \qquad \left[\frac{dv_1}{d\rho} - \frac{\sigma}{\rho} v_1\right]\rho = -0.92302 \tag{5.19}$$

The solution of problem (5.18), (5.19) can be obtained by a method analogous to the previous exponential series method.

In (5.18) and (5.19) it is necessary to perform the substitutions of the form (5.4) and to seek the solution of the problem in the interval $[0, b^2]$ in the form of (5.8), and in the interval $[b^2, 1]$ in the form $C_1(1 - x)$.

The constant C_1 is determined from the condition of the continuity of v_1, together with its derivative, according to Theorem 2.2. The value a_0 is found as the solution of the linear algebraic equation. The graphs of the functions $v_0 + \varepsilon v_1$, $u_0 + \varepsilon u_1$, $w_0 + \varepsilon w_1$ are also represented in Figs. 1, 2 and 3, and are marked with the number 2.

Let us turn to the evaluation of (φ_1, ψ_1) which is the approximate solution of the problem (1.1), (1.2), (5.2) with the consideration of terms of order ε. For this we find g_1, η_1 for $s = 1$, from (2.6) and (2.7), and we substitute these and the previously calculated values of the functions v_0, u_0, g_0 etc. into (2.1). The value of the deflection we shall find according to Formula (5.17), but with the substitution of u_0 by ψ_1. The approximate solution of the problem is represented in the graphs in Figs. 2, 3 and 4 and marked with the number 3. We note that in Fig. 2 the quantity φ_1 coincides with $v_0 + \varepsilon v_1$ and is correct up to values of the order ε^2; ψ_1 is a continuous function, changing rapidly in the neighborhood of the points $\rho = b$ and $\rho = 1$.

Finally, we calculate the bending moment arising in a plate. We have

$$M = -D\left(\frac{d^2w}{dr^2} + \frac{\sigma}{r}\frac{dw}{dr}\right), \qquad D = \frac{Eh^3}{12(1-\sigma^2)} \qquad (0 \leqslant r \leqslant a) \qquad (5.20)$$

Passing over to dimensionless variables we obtain

$$M_0 = -\frac{M}{Eha} = \varepsilon^2\left(\frac{du}{d\rho} + \frac{\sigma}{\rho}u\right)$$

In Fig.4, the graphic representation of the function $M_1 = M_0 \times 10^1$ (marked with the number 3) is given. It is interesting to note, that in the membrane the bending moments are equal to zero (in Fig.4 this is a straight line coinciding with the abscissa axis and marked with the number 1), and the extreme values of M_1 are found at points $\rho = b$, and $\rho = 1$.

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